

Griffiths phases in the contact process on complex networks

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Dynamical processes occurring on top of complex networks have become an exciting area of research. Quenched disorder plays a relevant role in general dynamical processes and phase transitions, but the effect of *topological quenched disorder* on the dynamics of complex networks has not been systematically studied so far. Here, we provide heuristic and numerical analyses of the contact process defined on some complex networks with topological disorder. We report on Griffiths phases and other rare region effects, leading rather generically to anomalously slow relaxation in generalized small-world networks. In particular, it is illustrated that Griffiths phases can emerge as the consequence of pure topological heterogeneity if the topological dimension of the network is finite.

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I. INTRODUCTION

The study of complex networks has been flourishing since the introduction of a simple model describing the emergence of scaling in random networks [1]. Multidisciplinary applications involve, for example, the World Wide Web, various biological, sociological and technological networks. These are, in many cases, scale-free networks (for recent reviews see [2],[3]). Other families of complex network models are those composed of a d -dimensional regular lattice and additional long edges [4]. These arise e.g. in sociophysics [3], in the context of conductive properties of linear polymers with cross-links that connect remote monomers [5], in public traffic systems [6], in the case of nanowires [7], and in many other examples. In general, a pair of nodes separated by the distance l is connected by an edge with the asymptotic probability for large l :

$$P(l) = \beta l^{-s} . \quad (1)$$

In the special case $s = 0$, the edges exist with a length-independent probability, as in small world networks, therefore we call the nets with generic s generalized small world networks (GSW). If $s \geq 2$, they are characterized by a finite topological dimension D , i.e. $N(l) \sim l^D$, where $N(l)$ is the number of nodes within a distance l from a given node. (Note that “small-world” property means that the number of neighbors of a node increases exponentially with the distance from it, i.e. formally $D = \infty$).

Dynamical processes defined on networks are of recent interest, for example in the context of optimization dynamics of spreading or transport processes [8]. Different dynamical processes, defined on regular lattices, often exhibit scaling behavior, that can be classified into basic universality classes [9]. Instead, the existence of scaling universality classes on network architectures is not clearly established. In many cases, the presence of short average distances induces mean-field type of (fast) dynamics. It is also well known that structural heterogeneity may lead to more complex behavior; for example, on scale-free networks, characterized by a power-law degree distribution $P(k) \propto k^{-\gamma}$ [1], critical exponents may vary with the degree exponent γ [2, 12]. In related ‘annealed networks’, where the links change rapidly, γ -dependent critical exponents have also been reported [10, 11]. Real networks are, however, ‘quenched’ in many cases, i.e. the topology changes slowly with respect to the dynamical process evolving on them.

In the context of the statistical physics of models defined on Euclidean lattices, it is well known that quenched randomness can generate “rare-region effects” in the so called *Griffiths phase* (GP), where, in many cases, algebraic scaling is observed (as opposed to pure systems in which power-laws are observed only at critical points) with scaling exponents changing continuously with the control parameter [13–16, 30]. Furthermore, at the phase transition the evolution becomes logarithmically slow, and it is controlled by ‘activated scaling’. A nontrivial question can be posed: Under which conditions do rare-region effects and GPs occur in network systems? Can they emerge out of topological disorder alone (i.e. without disorder in the transition rates)? These problems have not been systematically studied so far. At first sight one might guess that in network models the dynamics should be very fast, due to the strong inter-connectedness, i.e. information can propagate very efficiently throughout the system, resulting in mean-field like, exponential relaxation. Recently, however, generic slow dynamics has been reported to occur in various network models; its origin has been attributed to heterogeneity [17] or to local bursty activity patterns [18, 19]. To tackle

this type of problems, here we present a study of the contact process (CP) [20] (i.e. the simplest possible model for epidemic spreading, or information flow) on different GSW networks, including regular networks and non-regular ones.

II. NUMERICAL ANALYSES

We study the CP on random networks composed of a d -dimensional lattice and a set of long edges with unbounded length. Any pair of nodes, separated by the distance l , is connected by an edge with some l -dependent probability, Eq.(1) [4, 24, 25]. An intriguing feature of these type of graphs for $d = 1$ is that in the marginal case ($s = 2$) intrinsic properties show power-law behavior and the corresponding exponents vary continuously with the prefactor β . Indeed, the topological dimension D of such networks has been conjectured to depend on β [24]. Instead, for larger values of s long-ranged links are irrelevant, and $D(\beta) = 1$, while for $s < 2$ the topological dimension diverges. It has been claimed in a recent paper [28] that if $D(\beta)$ is finite, Griffiths phases and similar rare-region effects can appear.

We have considered the CP [20], in which each infected (active) sites is healed at rate 1, whereas each of its nearest-neighbor sites is infected at rate λ/k (where k is the degree of the site) so that the total infection rate is λ . In numerical simulations we update active sites randomly and increment the time by $1/N_a$, where N_a is the number of active sites. In this way, all active sites are updated on average once every time unit. For a critical infection rate λ_c we find a phase transition from the active to an absorbing phase [21], with vanishing density of infected sites.

We have performed numerical simulations initiated either from a fully active state or from a single active seed. In the former case, the density $\rho(t)$ of active sites has been measured.

A. Non-regular random networks

The precise definition of the networks we consider is as follows[23, 24]. We start with N nodes, numbered as $1, 2, \dots, N$ and define the distance between node i and j as $l = \min(|i - j|, N - |i - j|)$. We connect any pair of sites separated by a distance $l = 1$ (i.e. neighboring sites on the ring) with probability 1, and pairs with $l > 1$ with a probability $P(l) = 1 - \exp(-\beta l^{-s})$. This implies $P(l) = \beta l^{-s}$ at large distances. In the case $s > 2$, long-range links do not change the topological dimension, which remains unity, and hence the critical behavior is expected to be similar to that of the one-dimensional CP with *disordered transition rates*. In the latter model, the critical dynamics is logarithmically slow, for instance, starting from a fully active state, the density decays asymptotically as $\rho \propto (\ln(t))^{-\tilde{\alpha}}$ with $\tilde{\alpha} = 0.38197$ [16]. For $s = 3$ (and $\beta = 2$) we have run extremely long ($t_{max} = 2^{28}$ MCs) simulations on networks with number of nodes $N = 10^5$. Besides observing a GP, we have found that data are compatible with the above form of logarithmic dependence on time, and have located the critical point at $\lambda_c = 2.783(1)$. As Fig. 1(a) shows the assumption on activated scaling with $\tilde{\alpha} = 0.38197$ is satisfactory, although only after an extremely long crossover time to be discussed later [29].

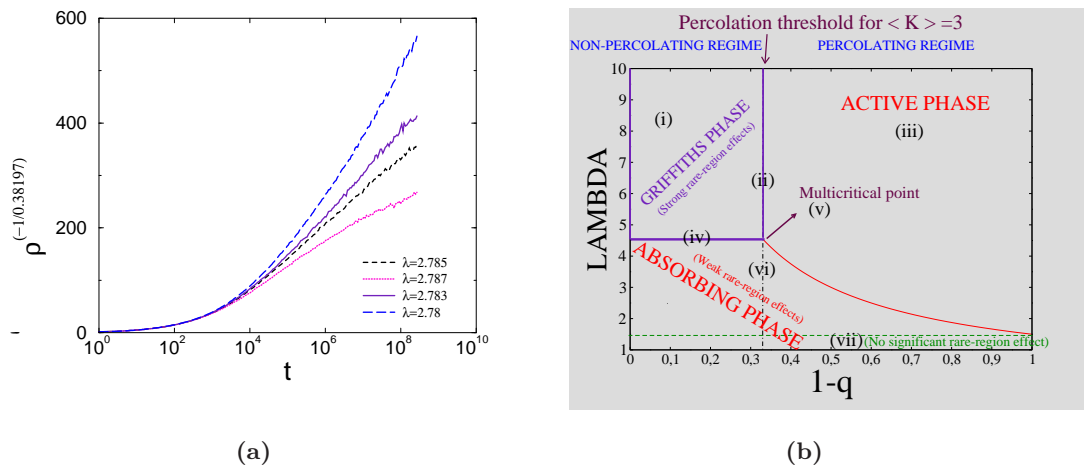


FIG. 1: Time-dependence of the density in the $s = 3$, $\beta = 2$ random network. Inset: local slopes of the same (lower curves), and local slopes of the survival probability in the 1d QCP simulations (dashed curves). Slow convergence of the effective exponents can be observed. (b) Phase diagram of QCP on the ER graph for $r = 0$.

For the marginal case $s = 2$, where the topological dimension is a continuous function of β , numerical simulations

indicate qualitatively different scenarios of the phase transition for different values of β . If β is small enough, GP is observed and the scaling at the transition is of logarithmic form. However, when β is large enough, the GP seems to be lacking and the critical dynamics follows a power-law. The latter type of behavior is observed also for $s < 2$, where formally $D = \infty$.

An “annealed” counterpart of CP on the above random networks is the 1d CP with Lévy flight distributed activation probabilities $P(r) \propto r^{-d-\sigma}$. In this model the dynamical exponents are known to depend continuously on σ [27]. The estimated dynamical exponent of the CP on the above random networks in case of power-law critical behavior is found to be compatible with the dynamical exponent of the Lévy-flight CP with $d = 1$. For further details the reader is referred to [28, 29].

B. 3-regular (or cubic) random networks

In the networks studied so far the degree of nodes was random. In the following we consider networks with a “weaker” topological disorder, in the sense that the degree of nodes is constant (3). Such random networks with nodes of degree 3 can be constructed in the following way [25]. A one-dimensional periodic lattice with N sites is given, where the degree of all sites is initially 2. Sites of degree 2 are called “free sites”. Let us assume that N is even and k is a fixed positive integer. A pair of free sites is selected such that the number of free sites between them is $k - 1$ (the number of non-free nodes can take any value) and this pair is then connected by a link. That means, for $k = 1$, neighboring free sites are connected, for $k = 2$ next-to-neighboring ones, etc. This step, which raises to 3 the degree of two free sites, is then iterated until $2(k - 1)$ free sites are left. These are then paired in an arbitrary way, which does not affect the properties of the network in the limit $N \rightarrow \infty$. In the resulting network, all sites are of degree 3, and one can show that the probability of edges is given by Eq. (1) with $s = 2$. For certain networks of this type it has been demonstrated that the long-ranged connections result in a finite topological dimension which is less than one [25]. Furthermore, in the case of aperiodic networks, the critical exponents of the CP depend on the underlying aperiodic modulation [26]. In the following, we concentrate on random networks in which the pairs to be connected are selected randomly (with equal probability) in the above procedure for a fixed k . In this case, one can show that the prefactor in Eq. (1) is given by $\beta = k/2$.

We have performed numerical simulations for the CP on $k = 1$ random networks with $N = 10^7$ nodes. The averaging was done over 200 different network realizations for each λ value and the maximum time is $t_{max} \leq 2^{26}$. As Fig. 2(a) shows, power-law decay with continuously changing exponent emerges for a range of λ -s. By analyzing the effective decay exponents, defined as the local slopes of the density $\alpha_{\text{eff}}(t) = -(\ln \rho(t) - \ln \rho(t'))/(\ln(t) - \ln(t'))$, where $t/t' = 2$, the curves do not level-off for large times, instead one can observe a small drift. One can easily derive that the functional dependence of the effective exponent $\alpha_{\text{eff}}(t) = \alpha + x/\ln(1/t)$ is related to the scaling correction $\rho(t) = t^{-\alpha} \ln^x(1/t)$. By plotting $\alpha_{\text{eff}}(t)$ on logarithmic time scales (see inset of Fig. 2(a)) it turns out that the drift describes logarithmic corrections to power-laws and that α is a non-universal quantity which depends on λ . The possibility of logarithmic corrections has already been pointed out in the case of CP with quenched disorder [15].

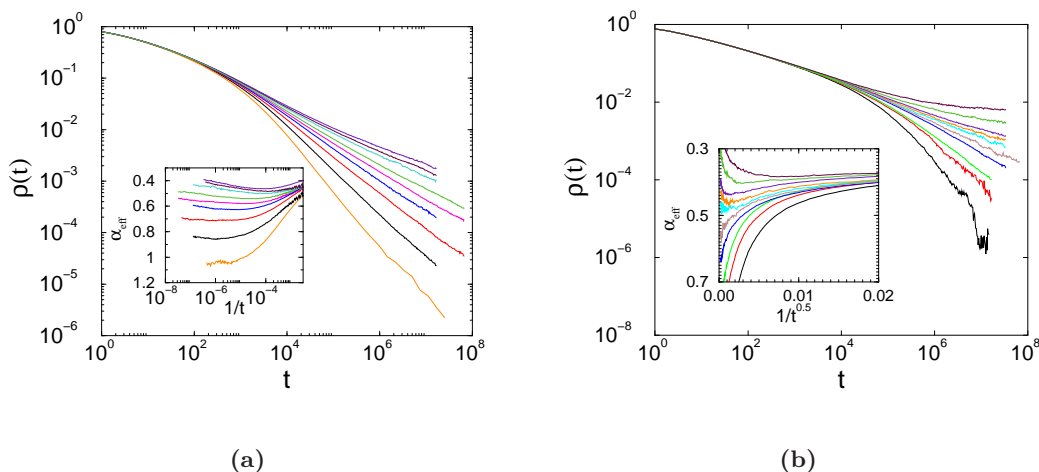


FIG. 2: Density decay in 3-regular, random networks. (a) GP in $k = 1$ between $\lambda = 2.535$ and $\lambda = 2.57$ (from bottom to top). (b) Critical point for $k = 2$ between $\lambda = 2.156$ and $\lambda = 2.1598$ (from bottom to top). The inset shows the corresponding local slopes.

In accordance with results on non-regular networks with $s = 2$, the phase transition is qualitatively different for

$k = 2$ which corresponds to a higher value of β . Here, a standard phase transition seems to appear at $\lambda_c = 2.15835(5)$, with the density decay exponent $\alpha \simeq 0.51(5)$ (Fig. 2(b)). This value agrees again with that of the Lévy flight CP with $d = 1$ and $d + \sigma = 2$.

III. QCP ON ERDŐS-RÉNYI NETWORKS

Apart from studying the effect of heterogeneous (i.e. disordered) topologies on the dynamics of the contact process, we have also scrutinized the behavior of the disordered contact process on random networks. In particular, we have considered CP on ER graphs [22] with a quenched disordered infection rate: a fraction $1 - q$ of the nodes (type-I) take value λ and the remaining fraction q (type-II nodes) take a reduced value λr , with $0 \leq r < 1$. Pair mean-field approximations [28, 29] lead to the following critical threshold

$$\lambda_c(q) = \frac{\langle k \rangle}{\langle k \rangle - 1} \frac{1}{1 - q}. \quad (2)$$

Type-I nodes experience a percolation transition, where the type I-to-type I average degree is 1, i.e. at $q_{perc} = 1 - \langle k \rangle^{-1}$. For $q > q_{perc}$ activity cannot be sustained: type-I clusters are finite and type-II ones do not propagate activity.

Numerical simulations and optimal fluctuation theory for $\langle k \rangle = 3$ ($q_{perc} = 2/3$) leads to the complex phase diagram shown on Fig. 1(b). In agreement with (2) one finds a critical active/absorbing phase transition, but below the percolation threshold the absorbing phase splits into parts. In particular, for $\lambda \gtrsim 4.5$ (iv) rare, percolating regions may occur, which cause power-law dynamics (i), i.e. a Griffiths phase, compared to the inactive phase of simple CP, which is purely exponential. Further details can be found in [28, 29].

IV. CONCLUSIONS

We have illustrated that, besides quenched site disorder, topological disorder by itself can result in slow dynamics and non-universality, at least for the contact process. We expect these results to have a broad spectrum of implications for propagation phenomena and other dynamical process taking place on networks, and to be relevant for the analysis of both models and empirical data [17]. We have claimed in a different publication [28] that having finite topological dimension is a necessary condition for the occurrence of slow, GP type of evolution in complex networks, at least for CP type of dynamics. Investigation of other factors promises to be an interesting, open field of research.

Acknowledgments

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